Simplification of the tractive energy demand formula for a simple driving profile

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1 Introduction

This document present the steps taken for an analytical integration of the tractive energy demand formula for a simplified speed profile (constant acceleration, constant speed and constant deceleration).

1.1 Simplified speed profile

Figure 1 shows the simplified speed profile which is assumed:

![Simplified speed profile](image)

(a) Case 1

(b) Case 2

Figure 1: Simplified driving profile

1.2 General tractive energy demand formula

\[
E_{a(t)>0} = \frac{1}{\eta_r \eta_{PE} \eta_m} \int \left( Mgf \cos \alpha + Mg \sin \alpha + \frac{\rho C_d A}{2} v(t)^2 + \delta Ma(t) \right) v(t) \, dt \tag{1}
\]

\[
E_{a(t)<0} = r_{reg} \eta_r \eta_{PE} \eta_m \int \left( Mgf \cos \alpha + Mg \sin \alpha + \frac{\rho C_d A}{2} v(t)^2 + \delta Ma(t) \right) v(t) \, dt \tag{2}
\]
The following table describes the constants used:

<table>
<thead>
<tr>
<th>Constant</th>
<th>Denoted Variable</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight of the vehicle</td>
<td>$M$</td>
<td>kg</td>
</tr>
<tr>
<td>Gravitational acceleration</td>
<td>$g$</td>
<td>m s$^{-2}$</td>
</tr>
<tr>
<td>Rolling resistance</td>
<td>$f$</td>
<td>-</td>
</tr>
<tr>
<td>Air density at 25°C</td>
<td>$\rho$</td>
<td>kg m$^{-3}$</td>
</tr>
<tr>
<td>Air resistance coefficient</td>
<td>$C_d$</td>
<td>-</td>
</tr>
<tr>
<td>Cross-section area</td>
<td>$A$</td>
<td>m$^2$</td>
</tr>
<tr>
<td>Average efficiency of transmission</td>
<td>$\eta_t$</td>
<td>-</td>
</tr>
<tr>
<td>Average efficiency of inverter</td>
<td>$\eta_{PE}$</td>
<td>-</td>
</tr>
<tr>
<td>Average efficiency of motor</td>
<td>$\eta_m$</td>
<td>-</td>
</tr>
<tr>
<td>Gradient of the road</td>
<td>$\alpha$</td>
<td>-</td>
</tr>
<tr>
<td>Rotational inertia factor</td>
<td>$\delta$</td>
<td>-</td>
</tr>
</tbody>
</table>

In the following, let $K_d = 0.5 \rho C_d A$. Moreover, $\eta$ is defined such that:

$$\eta = \begin{cases} 
\frac{1}{\eta_t \eta_{PE} \eta_m} & \text{for } a(t) \geq 0 \\
\frac{1}{r_{reg} \eta_t \eta_{PE} \eta_m} & \text{for } a(t) < 0
\end{cases}$$

(3)

## 2 Assumptions concerning acceleration and speed

### 2.1 Summary of the input parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Denoted Variable</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acceleration rate</td>
<td>$a_+$</td>
<td>m s$^{-2}$</td>
</tr>
<tr>
<td>Deceleration rate</td>
<td>$a_-$</td>
<td>m s$^{-2}$</td>
</tr>
<tr>
<td>Coasting speed</td>
<td>$v_1$</td>
<td>m s$^{-1}$</td>
</tr>
<tr>
<td>Total distance travelled</td>
<td>$D$</td>
<td>m</td>
</tr>
</tbody>
</table>

### 2.2 Explanation and derivation of the expressions for the parameters

Let $D$ be the total distance driven, with $d_0$ the accelerating distance between $t_0$ and $t_1$, $d_1$ the constant speed distance between $t_1$ and $t_2$ and $d_2$ the decelerating distance between $t_2$ and $t_3$. Thus $D = d_0 + d_1 + d_2$.

#### 2.2.1 Case 1: $D$ is large enough to reach the constant speed phase

Please refer to Figure 1a.

A constant acceleration value of $a_+$ is assumed between $t_0$ and $t_1$ (initial speed $v_0 = 0$):

$$a(t) = a_+, \quad v(t) = a_+ t, \quad x(t) = \frac{a_+ t^2}{2}, \quad t_0 \leq t < t_1$$

(4)
A constant speed of value \( v_1 \) is assumed between \( t_1 \) and \( t_2 \):

\[
a(t) = 0, \quad v(t) = v_1, \quad x(t) = x(t_1) + v_1 t, \quad t_1 \leq t < t_2
\]  \( (5) \)

A constant deceleration of value \( a_- \) is assumed between \( t_2 \) and \( t_3 \) (\( a_- \) is defined as having a negative value, for example \( a_- = -1 \text{ m/s} \)):

\[
a(t) = a_-, \quad v(t) = v_1 + a_- (t - t_2), \quad t_2 \leq t < t_3
\]  \( (6) \)

We can calculate the time needed to reach the plateau \( (t_1) \) and the corresponding accelerating distance \( (d_0) \):

\[
v_1 = v(t_1) = a_+ t_1 \quad \Rightarrow \quad t_1 = \frac{v_1}{a_+}
\]  \( (7) \)

\[
d_0 = x(t_1) = \frac{a_+ t_1^2}{2} = \frac{a_+ v_1^2}{2a_+} = \frac{v_1^2}{2a_+}
\]  \( (8) \)

In the same way we arrive at the decelerating distance \( (d_2) \):

\[
d_2 = -\frac{v_1^2}{2a_-}
\]  \( (9) \)

This lets us calculate the length of the portion with a constant speed \( (d_1) \) as a function of the chosen parameters:

\[
D = d_0 + d_1 + d_2 \quad \Rightarrow \quad d_1 = D - (d_0 + d_2) = D - \frac{v_1^2}{2} \left( \frac{1}{a_+} - \frac{1}{a_-} \right)
\]  \( (10) \)

**Summary for case 1:**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Denoted Variable</th>
<th>Unit</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acceleration distance</td>
<td>( d_0 )</td>
<td>m</td>
<td>( \frac{v_1^2}{2a_+} )</td>
</tr>
<tr>
<td>Constant speed distance</td>
<td>( d_1 )</td>
<td>m</td>
<td>( D - \frac{v_1^2}{2} \left( \frac{1}{a_+} - \frac{1}{a_-} \right) )</td>
</tr>
<tr>
<td>Deceleration distance</td>
<td>( d_2 )</td>
<td>m</td>
<td>( -\frac{v_1^2}{2a_-} )</td>
</tr>
</tbody>
</table>

**2.2.2 Case 2:** \( D \) is not large enough to reach the constant speed phase

Please refer to Figure 1b.

The total distance and the chosen acceleration and deceleration rates do not allow the vehicle to reach the desired coasting speed. In this case:

\[
d_1 = 0
\]  \( (11) \)

\( \Delta t_0 \) is defined as the duration of the acceleration, \( \Delta t_2 \) as the duration of the deceleration, and \( v_{\text{max}} \) as the maximum velocity reached at the point where the vehicle transitions from accelerating to decelerating.
We have the following relationships:

\[
\begin{align*}
\{ \quad a_+ &= \frac{v_{\text{max}}}{\Delta t_0} \\ -a_- &= \frac{v_{\text{max}}}{\Delta t_2} \quad \Rightarrow \quad a_+ \Delta t_0 = -a_- \Delta t_2 \quad (= v_{\text{max}}) \quad \Rightarrow \quad \Delta t_2 = -\frac{a_+}{a_-} \Delta t_0 & (12) \\
\end{align*}
\]

Moreover, based on (8), we have \( d_0 = \frac{a_+ \Delta t_0^2}{2} \) and similarly \( d_2 = -\frac{a_- \Delta t_2^2}{2} \). It follows that:

\[
D = d_0 + d_2 = d_0 - \frac{a_- \Delta t_2^2}{2} \quad (13)
\]

Replacing \( \Delta t_2 \) with the expression from (12):

\[
D = d_0 - \frac{a_-}{2} \left( -\frac{a_+}{a_-} \Delta t_0 \right)^2 \quad (14)
\]

\[
= d_0 - \frac{a_+^2 \Delta t_0^2}{2a_-} \quad (15)
\]

\[
= d_0 - \frac{a_+ \Delta t_0^2}{a_-} \quad (16)
\]

\[
= d_0 - \frac{a_+}{a_-} d_0 \quad (17)
\]

\[
= d_0 \left( 1 - \frac{a_+}{a_-} \right) \quad (18)
\]

Finally we obtain:

\[
d_0 = \frac{D}{1 - \frac{a_+}{a_-}} = D \left( \frac{-a_-}{a_+ - a_-} \right) \quad (19)
\]

Note that since \( a_- \leq 0 \), the term \( -\frac{a_+}{a_-} \) is positive and thus the denominator is always greater than or equal to one.

\[
d_2 = D - d_0 = \frac{D}{1 - \frac{a_+}{a_-}} = D \left( \frac{a_+}{a_+ - a_-} \right) \quad (20)
\]

**Summary for case 2:**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Denoted Variable</th>
<th>Unit</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acceleration distance</td>
<td>( d_0 )</td>
<td>m</td>
<td>( \frac{D}{1 - \frac{a_+}{a_-}} )</td>
</tr>
<tr>
<td>Constant speed distance</td>
<td>( d_1 )</td>
<td>m</td>
<td>0</td>
</tr>
<tr>
<td>Deceleration distance</td>
<td>( d_2 )</td>
<td>m</td>
<td>( \frac{D}{1 - \frac{a_+}{a_-}} )</td>
</tr>
</tbody>
</table>
3 Introduction of the assumptions in the general formula and integration

Note: the following results are valid in both cases detailed in sections 2.2.1 and 2.2.2. The only difference between them is how the values of \(d_0\), \(d_1\) and \(d_2\) are calculated.

3.1 Constant speed phase

We begin with the constant speed phase, which is the easiest to integrate. From (1) and (5) we derive:

\[
E_{a(t)=0} = \eta \int_{t_1}^{t_2} \left( Mg \cos \alpha + Mg \sin \alpha + K_d v_1^2 \right) v_1 \, dt \quad \text{since } a(t) = 0 \tag{21}
\]

\[
= \eta \left( Mg \cos \alpha + Mg \sin \alpha + K_d v_1^2 \right) v_1 \left( t_2 - t_1 \right) \tag{22}
\]

\[
E_{a(t)=0} = \eta \left( Mg \cos \alpha + Mg \sin \alpha + K_d v_1^2 \right) d_1 \quad \text{since } d_1 = v_1 \left( t_2 - t_1 \right) \tag{23}
\]

3.2 Acceleration phase

For the acceleration phase, using (1) and (4):

\[
E_{a(t)=a_+} = \eta \int_{t_0}^{t_1} \left( Mg \cos \alpha + Mg \sin \alpha + \delta Ma_+ \right) a_+ t \, dt \tag{24}
\]

\[
= \eta a_+ \left( Mg \cos \alpha + Mg \sin \alpha + \delta Ma_+ \right) \int_{t_0}^{t_1} t \, dt + K_d a_+^2 \int_{t_0}^{t_1} t^3 \, dt \tag{25}
\]

\[
= \eta a_+ \left( Mg \cos \alpha + Mg \sin \alpha + \delta Ma_+ \right) \left[ \frac{t^2}{2} \right]_{t_0}^{t_1} + K_d a_+^2 \left[ \frac{t^4}{4} \right]_{t_0}^{t_1} \tag{26}
\]

\[
= \eta a_+ \left( Mg \cos \alpha + Mg \sin \alpha + \delta Ma_+ \right) \left( \frac{t_1^2 - t_0^2}{2} \right) + K_d a_+^2 \left( \frac{t_1^4 - t_0^4}{4} \right) \tag{27}
\]

\[
= \eta \frac{a_+}{2} \left( Mg \cos \alpha + Mg \sin \alpha + \delta Ma_+ \right) t_1^2 + K_d a_+^2 \left( \frac{t_1^4}{4} \right) \quad \text{since } t_0 = 0 \tag{28}
\]

\[
= \eta \frac{a_+ t_1^2}{2} \left( Mg \cos \alpha + Mg \sin \alpha + \delta Ma_+ + K_d a_+^2 \frac{t_1^2}{2} \right) \tag{29}
\]

\[
E_{a(t)=a_+} = \eta d_0 \left( Mg \cos \alpha + Mg \sin \alpha + \delta Ma_+ + K_d a_+ d_0 \right) \quad \text{since } d_0 = \frac{a_+ t_1^2}{2} \tag{30}
\]
3.3 Deceleration phase

For the deceleration phase, using (2) and (6):

\[ E_{a(t)=a_-} = \eta \int_{t_2}^{t_3} \left( Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_- + K_d (v_1 + a_- (t-t_2))^2 \right) (v_1 + a_- (t-t_2)) \, dt \]

(31)

Integration by substitution:

\[ u = a_- (t - t_2) \quad \quad t = \frac{u}{a_-} + t_2 \]

(32)

\[ du = a_- \, dt \quad \quad dt = \frac{du}{a_-} \]

(33)

\[ E_{a(t)=a_-} = \eta \int_{a_-(t_2-t_2)=0}^{a_-(t_3-t_2)} \left( Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_- + K_d (v_1 + u)^2 \right) (v_1 + u) \frac{1}{a_-} \, du \]

(34)

\[ = \eta \frac{1}{a_-} \left( Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_- \right) \int_{0}^{a_-(t_3-t_2)} (v_1 + u) \, du \]

(35)

\[ + K_d \int_{0}^{a_-(t_3-t_2)} (v_1 + u)^3 \, du \]

\[ = \eta \frac{1}{a_-} \left( Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_- \right) \left[ v_1 u + \frac{u^2}{2} \right]_{0}^{a_-(t_3-t_2)} \]

(36)

\[ + K_d \left[ \left( \frac{(v_1 + u)^4}{4} \right)_{0}^{a_-(t_3-t_2)} \right] \]

Note that \( v_1 + a_- (t_3 - t_2) = v(t_3) = 0 \) by definition of \( t_3 \). This also implies \( t_3 - t_2 = -\frac{v_1}{a_-} \).
\[ E_{a(t)=a_-} = \eta \left[ (Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_-) \left( v_1(t_3 - t_2) + \frac{a_- (t_3 - t_2)^2}{2} \right) + K_d \left( -\frac{v_1^4}{4a_-} \right) \right] \] (38)

\[ = \eta \left[ (Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_-) \left( -\frac{v_1^2}{a_-} + \frac{v_1^2}{2a_-} \right) + K_d \left( -\frac{v_1^4}{4a_-} \right) \right] \] (39)

\[ = \eta \left[ (Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_-) \left( -\frac{v_1^2}{2a_-} \right) + K_d \left( -\frac{v_1^4}{4a_-} \right) \right] \] (40)

\[ = \eta \left( -\frac{v_1^2}{2a_-} \right) \left[ (Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_-) + K_d \frac{v_1^2}{2} \right] \] (41)

Finally, using (9), and noting that \( \frac{v_1^2}{2} = -a_-d_2^2 \):

\[ E_{a(t)=a_-} = \eta \mathbf{d}_2 (Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_- - K_d a_- \mathbf{d}_2) \] (42)

The results is similar to (30). Since \( d_2 \geq 0 \) and \( a_- \leq 0 \), the term associated with \( \delta Ma_- \) is negative and corresponds to the energy recuperation from regenerative braking, while the terms involving \( Mgf \cos \alpha \) and \( K_d \) are positive, thus corresponding to the losses of rolling friction and air drag that cannot be recuperated.

### 4 Conclusion

Assuming the simple driving profile as shown in Figure 1, the total energy consumption can be calculated as follows:

\[ E_{total} = E_{a(t)=a_+} + E_{a(t)=0} + E_{a(t)=a_-} \] (43)

with

\[ E_{a(t)=a_+} = \eta \mathbf{d}_0 (Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_+ + K_d a_+ \mathbf{d}_0) \] (44)

\[ E_{a(t)=0} = \eta \mathbf{d}_1 (Mgf \cos \alpha + Mg \sin \alpha + K_d v_1^2) \] (45)

\[ E_{a(t)=a_-} = \eta \mathbf{d}_2 (Mgf \cos \alpha + Mg \sin \alpha + \delta Ma_- - K_d a_- \mathbf{d}_2) \] (46)