Simplification of the tractive energy demand formula for a simple driving profile

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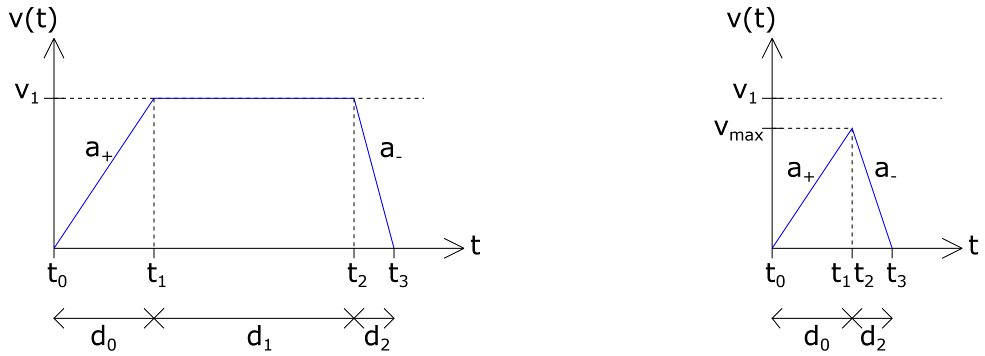
1 Introduction

This document present the steps taken for an analytical integration of the tractive energy demand formula for a simplified speed profile (constant acceleration, constant speed and constant deceleration).

1.1 Simplified speed profile

Figure 1 shows the simplified speed profile which is assumed:



(a) Case 1 (b) Case 2

Figure 1: Simplified driving profile

1.2 General tractive energy demand formula

$$E_{a(t)\geqslant 0} = \frac{1}{\eta_\mathrm{t}\,\eta_\mathrm{PE}\,\eta_\mathrm{m}} \int \left(Mgf\cos\alpha + Mg\sin\alpha + \frac{\rho\,C_\mathrm{d}\,A}{2} v(t)^2 + \delta M a(t) \right) v(t)\,\mathrm{d}t \quad (1)$$

$$E_{a(t)<0} = r_\mathrm{reg}\,\eta_\mathrm{t}\,\eta_\mathrm{PE}\,\eta_\mathrm{m} \int \left(Mgf\cos\alpha + Mg\sin\alpha + \frac{\rho\,C_\mathrm{d}\,A}{2} v(t)^2 + \delta M a(t) \right) v(t)\,\mathrm{d}t \quad (2)$$

The following table describes the constants used:

Constant	Denoted Variable	Unit
Weight of the vehicle	M	kg
Gravitational acceleration	g	$\mathrm{m\,s^{-2}}$
Rolling resistance	f	-
Air density at 25°C	ρ	$\mathrm{kg\,m^{-3}}$
Air resistance coefficient	C_{d}	-
Cross-section area	A	$\mathrm{m^2}$
Average efficiency of transmission	η_{t}	-
Average efficiency of inverter	η_{PE}	-
Average efficiency of motor	η_{m}	-
Gradient of the road	α	-
Rotational inertia factor	δ	-

In the following, let $K_{\mathrm{d}} = 0.5\,\rho\,C_{\mathrm{d}}\,A$. Moreover, η is defined such that:

$$\eta = \begin{cases} \dfrac{1}{\eta_{\mathrm{t}}\,\eta_{\mathrm{PE}}\,\eta_{\mathrm{m}}} & \text{for } a(t) \geqslant 0 \\ r_{\mathrm{reg}}\,\eta_{\mathrm{t}}\,\eta_{\mathrm{PE}}\,\eta_{\mathrm{m}} & \text{for } a(t) < 0 \end{cases} \tag{3}$$

2 Assumptions concerning acceleration and speed

2.1 Summary of the input parameters

Parameter	Denoted Variable	Unit
Acceleration rate	$\mathbf{a}_{+}$	$\mathrm{m\,s^{-2}}$
Deceleration rate	$\mathbf{a}_{-}$	$\mathrm{m\,s^{-2}}$
Coasting speed	$\mathbf{v_1}$	$\mathrm{m\,s^{-1}}$
Total distance travelled	$\mathbf{D}$	m

2.2 Explanation and derivation of the expressions for the parameters

Let D be the total distance driven, with d_0 the accelerating distance between t_0 and t_1, d_1 the constant speed distance between t_1 and t_2 and d_2 the decelerating distance between t_2 and t_3. Thus $D = d_0 + d_1 + d_2$.

2.2.1 Case 1: D is large enough to reach the constant speed phase

Please refer to Figure 1a.

A constant acceleration value of a_+ is assumed between t_0 and t_1 (initial speed $v_0 = 0$):

$$a(t) = a_+\,, \qquad v(t) = a_+\,t\,, \qquad x(t) = \frac{a_+\,t^2}{2}\,, \qquad t_0 \leq t < t_1 \tag{4}$$

A constant speed of value v_1 is assumed between t_1 and t_2:

$$a(t) = 0\,, \qquad v(t) = v_1\,, \qquad x(t) = x(t_1) + v_1\,t\,, \qquad t_1 \le t < t_2 \tag{5}$$

A constant deceleration of value a_- is assumed between t_2 and t_3 (a_- is defined as having a negative value, for example $a_- = -1$ m/s):

$$a(t) = a_-\,, \qquad v(t) = v_1 + a_-\,(t - t_2)\,, \qquad t_2 \le t < t_3 \tag{6}$$

We can calculate the time needed to reach the plateau (t_1) and the corresponding accelerating distance (d_0):

$$v_1 = v(t_1) = a_+\,t_1 \qquad \Rightarrow \qquad t_1 = \frac{v_1}{a_+} \tag{7}$$

$$d_0 = x(t_1) = \frac{a_+\,t_1^2}{2} = \frac{a_+ v_1^2}{2a_+^2} = \frac{{v_1}^2}{2a_+} \tag{8}$$

In the same way we arrive at the decelerating distance (d_2):

$$d_2 = -\frac{{v_1}^2}{2a_-} \tag{9}$$

This lets us calculate the length of the portion with a constant speed (d_1) as a function of the chosen parameters:

$$D = d_0 + d_1 + d_2 \quad \Rightarrow \quad d_1 = D - (d_0 + d_2) = D - \frac{{v_1}^2}{2}\left(\frac{1}{a_+} - \frac{1}{a_-}\right) \tag{10}$$

Summary for case 1:

Variable	Denoted Variable	Unit	Expression
Acceleration distance	d_0	m	$\frac{v_1^2}{2a_+}$
Constant speed distance	d_1	m	$D - \frac{{v_1}^2}{2}\left(\frac{1}{a_+} - \frac{1}{a_-}\right)$
Deceleration distance	d_2	m	$-\frac{{v_1}^2}{2a_-}$

2.2.2 Case 2: D is not large enough to reach the constant speed phase

Please refer to Figure 1b.

The total distance and the chosen acceleration and deceleration rates do not allow the vehicle to reach the desired coasting speed. In this case:

$$d_1 = 0 \tag{11}$$

Δt_0 is defined as the duration of the acceleration, Δt_2 as the duration of the deceleration, and v_{max} as the maximum velocity reached at the point where the vehicle transitions from accelerating to decelerating.

We have the following relationships:

$$\begin{cases} a_+ & = \frac{v_{\max}}{\Delta t_0} \\ -a_- & = \frac{v_{\max}}{\Delta t_2} \end{cases} \Rightarrow \quad a_+\Delta t_0 = -a_-\Delta t_2 \quad (= v_{\max}) \quad \Rightarrow \quad \Delta t_2 = -\frac{a_+}{a_-}\Delta t_0 \tag{12}$$

Moreover, based on (8), we have $d_0 = \frac{a_+\Delta t_0^2}{2}$ and similarly $d_2 = -\frac{a_-\Delta t_2^2}{2}$. It follows that:

$$D = d_0 + d_2 = d_0 - \frac{a_-\Delta t_2^2}{2} \tag{13}$$

Replacing Δt_2 with the expression from (12):

$$D = d_0 - \frac{a_-}{2}\left(-\frac{a_+}{a_-}\Delta t_0\right)^2 \tag{14}$$

$$= d_0 - \frac{a_+^2\Delta t_0^2}{2a_-} \tag{15}$$

$$= d_0 - \frac{a_+}{a_-}\frac{a_+\Delta t_0^2}{2} \tag{16}$$

$$= d_0 - \frac{a_+}{a_-} d_0 \tag{17}$$

$$= d_0\left(1 - \frac{a_+}{a_-}\right) \tag{18}$$

Finally we obtain:

$$d_0 = \frac{D}{1 - \frac{a_+}{a_-}} = D\left(\frac{-a_-}{a_+ - a_-}\right) \tag{19}$$

Note that since $a_- \leq 0$, the term $-\frac{a_+}{a_-}$ is positive and thus the denominator is always greater than or equal to one.

$$d_2 = D - d_0 = \frac{D}{1 - \frac{a_-}{a_+}} = D\left(\frac{a_+}{a_+ - a_-}\right) \tag{20}$$

Summary for case 2:

Variable	Denoted Variable	Unit	Expression
Acceleration distance	d_0	m	$\frac{D}{1-\frac{a_+}{a_-}}$
Constant speed distance	d_1	m	0
Deceleration distance	d_2	m	$\frac{D}{1-\frac{a_-}{a_+}}$

3 Introduction of the assumptions in the general formula and integration

Note: the following results are valid in both cases detailed in sections 2.2.1 and 2.2.2. The only difference between them is how the values of d_0*,* d_1 *and* d_2 *are calculated.*

3.1 Constant speed phase

We begin with the constant speed phase, which is the easiest to integrate. From (1) and (5) we derive:

$$E_{a(t)=0} = \eta \int_{t_1}^{t_2} \left(Mgf\cos\alpha + Mg\sin\alpha + K_\mathrm{d}\, v_1^2\right) v_1\,\mathrm{d}t \qquad \text{since } a(t)=0 \tag{21}$$

$$= \eta \left(Mgf\cos\alpha + Mg\sin\alpha + K_\mathrm{d}\, v_1^2\right) v_1\;(t_2 - t_1) \tag{22}$$

$$\boxed{E_{a(t)=0} = \eta \left(Mgf\cos\alpha + Mg\sin\alpha + K_\mathrm{d}\, \mathbf{v_1}^2\right) \mathbf{d_1}} \qquad \text{since } d_1 = v_1\;(t_2 - t_1) \tag{23}$$

3.2 Acceleration phase

For the acceleration phase, using (1) and (4):

$$E_{a(t)=a_+} = \eta \int_{t_0}^{t_1} \left(Mgf\cos\alpha + Mg\sin\alpha + K_\mathrm{d}\,(a_+t)^2 + \delta M a_+\right) a_+ t\,\mathrm{d}t \tag{24}$$

$$= \eta\, a_+ \left((Mgf\cos\alpha + Mg\sin\alpha + \delta M a_+) \int_{t_0}^{t_1} t\,\mathrm{d}t + K_\mathrm{d}\, a_+^2 \int_{t_0}^{t_1} t^3\,\mathrm{d}t \right) \tag{25}$$

$$= \eta\, a_+ \left((Mgf\cos\alpha + Mg\sin\alpha + \delta M a_+) \left[\frac{t^2}{2}\right]_{t_0}^{t_1} + K_\mathrm{d}\, a_+^2 \left[\frac{t^4}{4}\right]_{t_0}^{t_1} \right) \tag{26}$$

$$= \eta\, a_+ \left((Mgf\cos\alpha + Mg\sin\alpha + \delta M a_+) \frac{(t_1^2 - t_0^2)}{2} + K_\mathrm{d}\, a_+^2 \frac{t_1^4 - t_0^4}{4} \right) \tag{27}$$

$$= \eta\, \frac{a_+}{2} \left((Mgf\cos\alpha + Mg\sin\alpha + \delta M a_+)\, {t_1}^2 + K_\mathrm{d}\, a_+^2 \frac{{t_1}^4}{2} \right) \qquad \text{since } t_0 = 0 \tag{28}$$

$$= \eta\, \frac{a_+ t_1^2}{2} \left(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_+ + K_\mathrm{d}\, a_+^2 \frac{{t_1}^2}{2} \right) \tag{29}$$

$$\boxed{E_{a(t)=a_+} = \eta\, \mathbf{d_0}\; \left(Mgf\cos\alpha + Mg\sin\alpha + \delta M \mathbf{a}_+ + K_\mathrm{d}\, \mathbf{a}_+\, \mathbf{d_0}\right)} \qquad \text{since } d_0 = \frac{a_+ t_1^2}{2} \tag{30}$$

3.3 Deceleration phase

For the deceleration phase, using (2) and (6):

$$E_{a(t)=a_-} = \eta \int_{t_2}^{t_3} \left(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_- + K_\mathrm{d}\left(v_1 + a_-(t-t_2)\right)^2 \right) \left(v_1 + a_-(t-t_2)\right) \mathrm{d}t \tag{31}$$

Integration by substitution:

$$u = a_-(t-t_2) \qquad t = \frac{u}{a_-} + t_2 \tag{32}$$

$$\mathrm{d}u = a_-\,\mathrm{d}t \qquad dt = \frac{\mathrm{d}u}{a_-} \tag{33}$$

$$E_{a(t)=a_-} = \eta \int_{a_-(t_2-t_2)=0}^{a_-(t_3-t_2)} \left(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_- + K_\mathrm{d}\left(v_1+u\right)^2 \right) \left(v_1+u\right) \frac{1}{a_-}\mathrm{d}u \tag{34}$$

$$= \eta \frac{1}{a_-} \Bigg(\left(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_- \right) \int_0^{a_-(t_3-t_2)} \left(v_1+u\right)\mathrm{d}u \tag{35}$$

$$+ K_\mathrm{d} \int_0^{a_-(t_3-t_2)} \left(v_1+u\right)^3 \mathrm{d}u \Bigg)$$

$$= \eta \frac{1}{a_-} \Bigg(\left(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_- \right) \left[v_1 u + \frac{u^2}{2} \right]_0^{a_-(t_3-t_2)} \tag{36}$$

$$+ K_\mathrm{d} \left[\frac{\left(v_1+u\right)^4}{4} \right]_0^{a_-(t_3-t_2)} \Bigg)$$

$$= \eta \frac{1}{a_-} \Bigg[\left(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_- \right) \left(v_1 a_-(t_3-t_2) + \frac{\left(a_-(t_3-t_2)\right)^2}{2} \right)$$

$$+ K_\mathrm{d} \left(\frac{\left(v_1 + a_-(t_3-t_2)\right)^4}{4} - \frac{v_1^4}{4} \right) \Bigg] \tag{37}$$

Note that $v_1 + a_-(t_3-t_2) = v(t_3) = 0$ by definition of t_3. This also implies $t_3 - t_2 = -\frac{v_1}{a_-}$.

$$E_{a(t)=a_-} = \eta \left[(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_-) \left(v_1(t_3 - t_2) + \frac{a_-(t_3-t_2)^2}{2} \right) + K_\mathrm{d} \left(-\frac{v_1^4}{4a_-} \right) \right] \tag{38}$$

$$= \eta \left[(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_-) \left(-\frac{v_1^2}{a_-} + \frac{v_1^2}{2a_-} \right) + K_\mathrm{d} \left(-\frac{v_1^4}{4a_-} \right) \right] \tag{39}$$

$$= \eta \left[(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_-) \left(-\frac{v_1^2}{2a_-} \right) + K_\mathrm{d} \left(-\frac{v_1^4}{4a_-} \right) \right] \tag{40}$$

$$= \eta \left(-\frac{v_1^2}{2a_-} \right) \left[(Mgf\cos\alpha + Mg\sin\alpha + \delta M a_-) + K_\mathrm{d} \frac{v_1^2}{2} \right] \tag{41}$$

Finally, using (9), and noting that $\frac{v_1^2}{2} = -a_- d_2$:

$$\boxed{E_{a(t)=a_-} = \eta \, \mathbf{d_2} \left(Mgf\cos\alpha + Mg\sin\alpha + \delta M \mathbf{a}_- - K_\mathrm{d} \, \mathbf{a}_- \mathbf{d_2} \right)} \tag{42}$$

The results is similar to (30). Since $d_2 \geq 0$ and $a_- \leq 0$, the term associated with $\delta M a_-$ is negative and corresponds to the energy recuperation from regenerative braking, while the terms involving $Mgf\cos\alpha$ and K_d are positive, thus corresponding to the losses of rolling friction and air drag that cannot be recuperated.

4 Conclusion

Assuming the simple driving profile as shown in Figure 1, the total energy consumption can be calculated as follows:

$$E_{total} = E_{a(t)=a_+} + E_{a(t)=0} + E_{a(t)=a_-} \tag{43}$$

with

$$E_{a(t)=a_+} = \eta \, \mathbf{d_0} \left(Mgf\cos\alpha + Mg\sin\alpha + \delta M \mathbf{a}_+ + K_\mathrm{d} \, \mathbf{a}_+ \mathbf{d_0} \right) \tag{44}$$

$$E_{a(t)=0} = \eta \, \mathbf{d_1} \left(Mgf\cos\alpha + Mg\sin\alpha + K_\mathrm{d} \, \mathbf{v_1}^2 \right) \tag{45}$$

$$E_{a(t)=a_-} = \eta \, \mathbf{d_2} \left(Mgf\cos\alpha + Mg\sin\alpha + \delta M \mathbf{a}_- - K_\mathrm{d} \, \mathbf{a}_- \mathbf{d_2} \right) \tag{46}$$